Journal of Survey Statistics and Methodology (2025) ***00**,* *1–21*

INFERENCE FOR THE QUANTILE RATIO INEQUALITY INDEX IN THE CONTEXT OF SURVEY DATA

SILVIA SCARPA *
MARIA ROSARIA FERRANTE
STEFAN SPERLICH

There exist many statistical indicators for measuring economic inequality. Most of them rely on distribution moments or focus on a few selected percentiles at the tails of the distribution. Recently, a so-called quantile ratio index has been introduced. It considers the entire distribution and measures the distance between the (economic) equi-distribution scenario and the average ratio of quantiles below the median to their symmetric counterparts above it. We present a finite population framework for estimating this index and its standard error under some complex sampling designs. Our estimator demonstrates high accuracy and precision, even with relatively small samples. Being solely based on quantiles, this index exhibits remarkable robustness, having limited sensitivity to anomalous values and highly skewed distributions. This is also shown by an analysis of its influence function.

KEYWORDS: Finite population; Income and wealth; Inequality indicators; Quantile ratio index; Survey data.

Silvia Scarpa is a Research fellow, Department of Economics, University of Modena and Reggio Emilia, Via Jacopo Berengario, 51, 41121, Modena, Italy. Maria Rosaria Ferrante is a Full Professor in the Department of Statistical Sciences, University of Bologna, Via delle Belle Arti, 41, 40126, Bologna, Italy. Stefan Sperlich is a Full Professor at the Geneva School of Economics and Management, University of Geneva, Uni Mail, Boulevard du Pont-d'Arve 40/1211, Geneva, Switzerland.
This work was supported by the European Union—NextGenerationEU, Mission 4, Component 2, in the framework of the GRINS—Growing Resilient, INclusive and Sustainable project (PNRR—M4C2—I1.3—PE00000018—CUP J33C22002910001) to Si. Sc., M. R. F. (the views and opinions expressed are solely those of the authors and do not necessarily reflect those of the European Union, nor can the European Union be held responsible for them) and by the project 200021–192345 of the Swiss National Science Foundation to St. Sp.
*Address correspondence to Silvia Scarpa, Department of Economics, University of Modena and Reggio Emilia, Via Jacopo Berengario, 51, 41121, Modena, Italy; E-mail: silvia.scarpa@unimore.it

https://doi.org/10.1093/jssam/smaf024

Statement of Significance

Many studies have explored statistical inference for indicators of economic inequality, which are vital tools for policymakers to monitor and raise awareness about this issue. This work studies in detail the calculus, estimation, and inference of a recently introduced inequality indicator: the quantile ratio index for finite populations when only survey data are available. The proposed design-based estimator for this index and its variance perform well with data collected by the sampling designs considered, demonstrating its reliability as an effective tool for monitoring inequality. We also introduce an estimator for its standard error.

1. INTRODUCTION

Currently, the coefficient proposed by Gini (1914) is the most widely used indicator of inequality, for example, of income. A major limitation is its sensitivity to large values in the long tails of the distribution, which are quite common in practice (Giorgi and Gigliarano 2017). However, reliable inequality measures are crucial for many reasons, such as a proper assessment of policy effectiveness or trust in official statistics. For distributions with high skewness, inequality indicators based on the mean are inappropriate, as they lack in robustness. In contrast, quantile-based indicators typically focus on the consideration of tails. For instance, the quintile share ratio and related indicators compare different portions of income in the tails or just a particular percentile ratio. They ignore, however, the middle class, which should be, together with the poor, the main focus of public policies (Gastwirth 2017). Estimators that mainly rely on the tails of a distribution are strongly affected by anomalously extreme observations and the small portions of the sample, which can result in a high sampling variance, as well as by the issue of non-reporting, which tends to occur more frequently in the upper tail.

Recently, Prendergast and Staudte (2018) proposed the quantile ratio index (QRI), based on an integral of quantile ratios. Their main argument in its favor was its (alleged) simplicity, but to date, it has attracted little attention. A likely reason is that, to our knowledge, a detailed statistical study of this indicator using survey data is missing. In this work, we therefore analyze the statistical characteristics of the QRI in the finite population framework and propose a complete estimation framework. The QRI considers the entire distribution, comparing any quantile below the median with its symmetric counterpart above the median. It does not require the prior choice of specific percentiles, does not depend on the distribution mean, and eschews the above-mentioned problems. Moreover, we show that it has several desirable statistical properties, such as a bounded influence function, which implies limited sensitivity to extreme values. Prendergast and Staudte (2018) introduced the QRI together with an estimator whose definitions were based on distributions referring to infinite (super-)populations. In practice, the income or wealth data available are collected by surveys of finite populations, which frequently involve the use of particular sampling designs and weights.

The present work aims to provide a complete methodology that works for both infinite (super-) and finite populations. We provide this unified framework based on nonparametric methods and therefore do not rely on parametric assumptions. The inferential techniques we introduce are specifically useful for statistical offices and for studies in which the population of interest is finite or of unknown distribution. We propose and compare estimators that can be used for a wide range of survey data and that facilitate domain-specific analysis. In order to do so, we first tackle the challenge of estimating quantiles in finite populations. Then, with the selected quantile estimator, a QRI estimator is constructed that is almost unbiased, even for small samples. Moreover, we derive its influence function to show further desirable properties, such as boundedness, which ensures a limited standard error. Finally, we construct an estimator for the variance of the QRI estimator using a rescaled bootstrap that also works under complex sampling designs. Altogether, we can conclude that the QRI with its estimates is a reliable measure of inequality in a variety of settings, including small samples and/or small populations. Note that some of the simulations are carried out with data from the 2017 European Union Survey on Income and Living Conditions (EU-SILC), considering Italian regions as target domains. Following Eurostat, we consider as a target variable the equivalized disposable income, that is, household income adjusted for size and composition.

Section 2 introduces the QRI as proposed by Prendergast and Staudte (2018), followed by the finite population formulation, its estimator(s), and influence function. We discuss in detail quantile estimators for complex survey data. Detailed results of simulations on their performance are deferred to the Supplementary Material. In Section 3, the properties of the QRI estimator(s) are explored in terms of reliability and accuracy through simulations based on synthetic and Italian EU-SILC data. Then we introduce sampling variance estimators. Section 4 concludes.

2. THE QUANTILE RATIO INDEX

2.1 A Nonparametric Inequality Indicator

Consider the problem of measuring income inequality. As its name says, the QRI is based on the ratio between symmetric quantiles. It compares for all $p \in (0,1)$ the $p/2$-th quantiles with the $(1-p/2)$-th ones. In other words, it measures the average relative distance between incomes below the median and their symmetric counterparts situated in the upper part of the distribution. In that sense, it is normalized by the median, not by the mean.

We start from the definition given in Prendergast and Staudte (2018). Let Y be a random variable that indicates positive income with an absolutely continuous cumulative distribution function (henceforth cdf) F, and its quantile function defined as $Q(p) = F^{-1}(p)$. For any $p \in (0,1)$ denote the ratio between symmetric quantiles as $R(p) = \frac{Q(p/2)}{Q(1-p/2)}$. Set $R(0) = 0$ and $R(1) = 1$ such that $R(p) \in [0,1]$ is monotonically increasing. In case of equi-distribution, we get $R(p) = 1\, \forall p \in [0,1]$. Then the QRI is defined for an infinite (super-) population $\mathcal{U}$ as

$$QRI = 1 - \int_0^1 R(p)dp = 1 - \int_0^1 \frac{Q(p/2)}{Q(1-p/2)}dp, \quad 0 \le QRI \le 1. \qquad (1)$$

This formulation reminds us of the index $\xi = 1 - \int_0^1 \frac{Q(p)}{Q^*(p)}dp$ introduced by Zenga (2007), where $Q^*(p) = I^{-1}(p)$ and $I(y) = \int_0^y \frac{t}{E(Y)}f(t)dt$. That one is not very popular, as it depends not only on the quantile function but also on the incomplete first moment function. The index recently proposed by Davydov and Greselin (2020) compares equal-sized groups from the opposite tails of the distribution, and it still involves the population mean. In contrast, the QRI does not depend on distribution moments. It could be interpreted as a nonparametric measure, but in an infinite population its value would be obtained exactly with a parametric distribution with a known quantile function. It exploits the properties of the ratios of quantiles, such as robustness, which were analyzed by Prendergast and Staudte (2019) and by Farcomeni and Geraci (2024) when additional auxiliary information was available.

The interpretation of the QRI is illustrated in figure 1. It measures the area between the equi-distribution line and the inequality curve $\{p, R(p)\}$. The closer the ratio is to 1, the more equally distributed is income. Yet, low values of $R(p)$ suggest that the distribution is highly skewed to the right. In figure 1, graphs $\{p, R(p)\}$ are shown for some distributions with known quantile functions. For example, given the same log-normal location parameter, increasing the scale parameter pushes the inequality curve away from the equi-distribution line.

The QRI is a relative inequality measure, invariant to proportional changes in incomes, respects the axioms of anonymity, median-preserving transfer, and population independence, in the sense that it does not depend on the number of people in the population owning the total income. These properties ensure that the QRI can be used to compare income distributions between regions and over time. Certainly, in case one uses a quantile estimator that strongly depends on the sample size, the QRI estimator indirectly depends on it too.

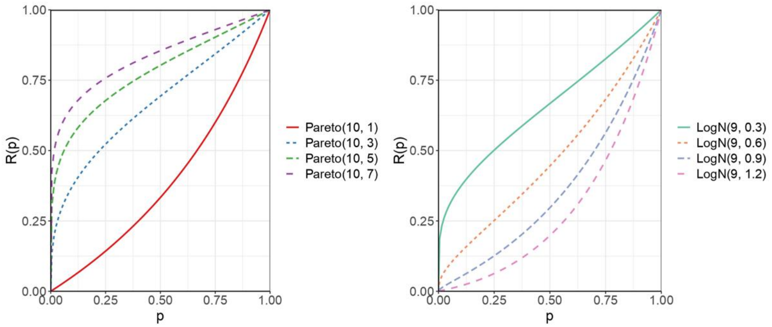

Figure 1. Examples of graph $\{p, R(p)\}$ for Pareto and Log-Normal distributions with different parameters.

2.2 The QRI in Finite Population

Socio-economic indicators are typically estimated by means of sample surveys, and the interest is in specific finite populations, say $U = \{1, \ldots, N\}$ of N identifiable units. Associated with each unit i is a value y_i of a certain characteristic, say positive income. We propose a definition of the QRI for such finite population U, extending ideas of Prendergast and Staudte (2018), who proposed an estimator of the QRI based on simple random sampling from infinite populations. Moreover, we develop a complete finite population framework, including estimators for the QRI and its standard deviation. We introduce nonparametric estimators that are equally valid for super-populations, certainly based on finite samples, and allow for complex sampling designs. More specifically, let us denote the cdf and quantile of order $p \in (0, 1)$ by

$$F_N(y) = \frac{1}{N}\sum_{i \in U} 1(y_i \leq y), \quad Q_N(p) = \inf\{y : F_N(y) \geq p\}, \tag{2}$$

where $1(\cdot)$ is the indicator function. Let us define a grid on $(0, 1)$ by $p_m = \frac{m-1/2}{M}$, for $m = 1, \ldots, M$. Suppose the population quantiles can be computed for each p_m as $Q_N(p_m)$. The ratio between symmetric quantiles is evaluated on the grid by $R_N(p)$, assuming that $R_N(0) = 0$ and $R_N(1) = 1$. A QRI approximation for U is then defined for any given grid by

$$QRI_N^M = \frac{1}{M}\sum_{m=1}^{M}(1 - R_N(p_m)) = \frac{1}{M}\sum_{m=1}^{M}\left(1 - \frac{Q_N(p_m/2)}{Q_N(1-p_m/2)}\right), \tag{3}$$

where $0 \leq QRI_N^M \leq 1$. This measure tells us how far the average ratio between symmetric quantiles evaluated on the grid in U is from the one of an equi-distribution. In our studies, M is set to 100, as it turned out in various intensive simulations that a larger M did not change our results.

When dealing with socio-economic aspects, data for the entire population are rarely available, and one relies on surveys. This is why our focus is on reliable estimators for the QRI, including estimators for standard errors. Imagine we draw a sample s of size n from the sampling space S by means of a complex sampling design $p(s) = \Pr(S = s)$, for all $s \subseteq U$. Denote by π_j the inclusion probability of unit j, that is $\Pr(j \in S) = \pi_j$, and by $\pi_{j,k} = \Pr(j,k \in S)$ the second-order inclusion probability, for all $j,k \in U$. The sampling weight w_j is the inversion of inclusion probability $w_j = \frac{1}{\pi_j}$ and may be calibrated on known marginal totals or adjusted for non-responses. We propose an estimator that can deal with data collected by such a sampling scheme. The estimator is given by (suppressing indices N, M)

$$\widehat{QRI} = \frac{1}{M}\sum_{m=1}^{M}\left(1 - \frac{\widehat{Q}(p_{m/2})}{\widehat{Q}(1-p_{m/2})}\right). \tag{4}$$

Obviously, the quantile estimators $\widehat{Q}(p)$ are the main drivers of the $\widehat{QRI}$ properties and performance. Even the sample size enters the estimator only via them.

2.3 Quantile Estimators for Finite Populations

There exist many definitions of sample quantiles, which provide nonparametric estimators of their population counterpart when the sampling weights are not considered. Unfortunately, different estimators can give quite different results for complex distributions and small samples (Dielman et al. 1994). Previous simulations studied this issue in detail: Parrish (1990) compared ten quantile estimators regarding bias and mean squared error (MSE) when small samples were drawn from a normal distribution. Dielman et al. (1994) extended their work by considering also skewed distributions. Hyndman and Fan (1996) made theoretical comparisons of nine sample quantile definitions, all implemented in statistical software.

However, less is known about quantile estimators when data are collected through a complex sampling design. Francisco (1987) and Francisco and Fuller (1991) provided the theoretical justification for using sample quantiles in stratified cluster sampling. Kreutzmann (2018) compared by simulations quantile estimators made for complex sampling designs available in common software. None of them provided conclusions that would help us to decide which estimator should be used for our $\widehat{QRI}$. Therefore, we extended the above studies to address the problem of estimating quantiles in finite populations and/or with complex designs.

First we introduce a general quantile estimation rule that allows us to consider several smoothed versions of the cdf, incorporates sampling weights, and can be applied to many complex sampling designs. Consider $Y_1, \ldots, Y_n$, sampled from population U, with $Y_{(1)}, \ldots, Y_{(n)}$ denoting their corresponding order statistics. Let $W_j = \sum_{i \in s} w_i \mathbb{1}(i \leq j)$ denote the cumulative sum of weights up to observation j. For $p = 0$ and $p = 1$, define $\widehat{Q}(0) = Y_{(1)}$ and $\widehat{Q}(1) = Y_{(n)}$. The p quantile estimator can be expressed as a weighted average of order statistics,

$$\widehat{Q}(p) = Y_{(k-1)} + \left(Y_{(k)} - Y_{(k-1)}\right)\left(\frac{p - \widehat{r}_{k-1}}{\widehat{r}_k - \widehat{r}_{k-1}}\right), \tag{5}$$

where $\widehat{r}_k$ indicates the estimator of the cdf, namely the plotting position, and the selected order k is such that $W_{k-1} - m_{k-1} < W_n p < W_k - m_k$, where m_k is determined by the interpolation method between adjacent data points. Linear interpolation between the points $(\widehat{r}, Y_{(k)})$ gives a quantile estimator for complex sampling data of the form (5). This rule works for different estimators $\widehat{r}_k$, does not require many theoretical assumptions, and can easily be implemented to obtain estimates of quantile-based inequality indicators.

We develop extensions of the continuous sample quantiles analyzed in Hyndman and Fan (1996), say $\widehat{Q}_l(p)$, $l = 4, \ldots, 9$, in order to account for sampling weights $w_1, \ldots, w_n$. Similar extensions were already considered by Lumley (2011) within his R package `survey`. However, we suggest several modifications to improve their efficiency and also fill the gap regarding documentation. Table 1 displays the estimators of the cdf, the interpolation parameters, and how the k-th order statistics are selected. Under simple random sampling, our extensions correspond to the ordinary sample quantiles reviewed by Hyndman and Fan (1996).

Kreutzmann (2018) proposed an extension of the Harrell and Davis (1982) estimator to complex survey data. We propose a further modification: starting from a known distribution of the expected value of the order statistics $\mathbb{E}[Y_{(W_n + w_n)p}]$, we redefine the weighting coefficient $\widehat{\mathcal{W}}_j(p)$, referred to the j-th unit in the sample, and obtain $\widehat{Q}_{HD}(p) = \sum_{j \in s} \widehat{\mathcal{W}}_j(p) Y_{(j)}$, where

$$\begin{aligned} \widehat{\mathcal{W}}_j(p) = {} & b_{(W_j/W_n)}\{(W_n + w_n)p, W_n - (W_n + w_n)p + w_n\} \\ & - b_{(W_{j-1}/W_n)}\{(W_n + w_n)p, W_n - (W_n + w_n)p + w_n\}. \end{aligned} \tag{6}$$

Eurostat recommends the Laeken quantile estimator for complex sampling data; see Alfons and Templ (2012), given by

$$\widehat{Q}_{Laek}(p) = \begin{cases} \frac{1}{2}\left(Y_{(k)} + Y_{(k+1)}\right) & \text{if} \quad \sum_{j=1}^{k} w_j = p \sum_{j=1}^{n} w_j, \\ Y_{(k+1)} \quad \text{if} & \sum_{j=1}^{k} w_j < p \sum_{j=1}^{n} w_j < \sum_{j=1}^{k+1} w_j. \end{cases} \tag{7}$$

Although this is a discontinuous function, it is included in our simulations and analysis, as it is widely used when studying income inequality.

Table 1. Extension of Hyndman and Fan (1996) Quantile Estimators Incorporating Sampling Weights

Estimator	$\widehat{\mathbf{r}}_{\mathbf{k}}$	$\widehat{\mathbf{m}}_{\mathbf{k}}$	$\mathbf{k}$
$\widehat{Q}_4(p)$	$\frac{W_k}{W_n}$	0	$W_{k-1} \leq W_n p < W_k$
$\widehat{Q}_5(p)$	$\frac{W_k - \frac{1}{2}w_k}{W_n}$	$\frac{w_k}{2}$	$W_{k-1} - \frac{w_{k-1}}{2} \leq W_n p < W_k - \frac{w_k}{2}$
$\widehat{Q}_6(p)$	$\frac{W_k}{W_n + w_n}$	$w_n p$	$W_{k-1} \leq (W_n + w_n)p < W_k$
$\widehat{Q}_7(p)$	$\frac{W_{k-1}}{W_{n-1}}$	$w_k - w_n p$	$W_{k-2} \leq W_{n-1} p < W_{k-1}$
$\widehat{Q}_8(p)$	$\frac{W_k - \frac{1}{3}w_k}{W_n + \frac{w_n}{3}}$	$\frac{w_k}{3} + \frac{w_n}{3}p$	$W_{k-1} - \frac{w_{k-1}}{3} \leq (W_n - \frac{w_n}{3})p < W_k - \frac{w_k}{3}$
$\widehat{Q}_9(p)$	$\frac{W_k - \frac{3}{8}w_k}{W_n + \frac{1}{4}w_n}$	$\frac{3}{8}w_k + \frac{w_n}{4}p$	$W_{k-1} - \frac{3w_{k-1}}{8} \leq (W_n + \frac{w_n}{4})p < W_k - \frac{3w_k}{8}$

These quantile estimators strongly depend on sample size and the tails of the underlying distribution and are expected to exhibit quite different behaviors. We performed extensive simulations with synthetic and real data (based on the SILC data set), generated as described in the next sections. Details on the study and its findings are provided in the Supplementary Material. Here we summarize the main conclusions: there is no quantile estimator that uniformly outperforms the others. The mean squared error is quite similar among the considered estimators, but $\widehat{Q}_4(p)$, which uses for the theoretical cdf the Hájek estimator, seems to be the less biased, as long as the sample size is not too small. As expected, when the sample size is very small, the considered quantile estimators are generally not reliable for extreme percentiles. The study indicates important differences, insinuating the importance of a proper quantile estimator selection for the performance of $\widehat{QRI}$. But unfortunately there is no clear winner, such that we have to study the QRI estimators constructed with the different quantile estimators.

2.4 The Influence Function of the QRI Estimator

The influence function is a statistical tool, initially proposed by Hampel (1974) to study the robustness of an estimator, and explored by Cowell and Victoria-Feser (1996) to study how inequality indicator estimators deal with an infinitesimal amount of contamination. They also analyzed it as a tool for a first-order approximation of the bias of an estimator. Estimators with a bounded influence function are clearly preferable. Otherwise, an important, potentially even unbounded bias could be caused by just a few observations. Similar arguments apply to the estimator's variance. Deville (1999) extended this conceptualization to the finite population framework. He considered a finite positive and discrete measure P, which puts unit mass if subject i is in the finite population U such that the total mass equals the population size N. In practice it may refer to a sample s (of U) of size n. Consider now a statistic of interest T (e.g. a quantile or the QRI) as a function of measure P, and let T be homogeneous, such that there exists a positive real number α with $T(tP) = t^{\alpha}T(P)$, for all $t \in \mathbb{R}_+$. Then the influence function of $T(P)$ for any $\varepsilon > 0$ is defined as its Gâteaux derivative in the direction of the Dirac measure at point y_i, δ_i. This can be written as

$$I(T(P))_i = \lim_{\varepsilon \to 0} \frac{T(P + \varepsilon\delta_i) - T(P)}{\varepsilon} \tag{8}$$

when the limit exists. Given a random sample s of size n, selected from U by means of a certain sampling design, a measure P is estimated by $\widehat{P}$ which allocates mass equal to the sampling weight w_i to any point y_i in the sample, having total mass $\widehat{N} = \sum_{j \in s} w_j$. $I(T(P))_i$ gives the linearized variable z_i of $T(\widehat{P})$, which allows the following approximation

$$T(\widehat{P}) - T(P) \approx \sum_{j \in s} w_j z_j - \sum_{i \in U} z_i \, .$$

Obviously, the variance of $T(\widehat{P})$ can be approximated by the one of $Z = \sum_{j \in s} w_j z_j$. As will be seen next, z_i often depends on unknown parameters that have to be estimated.

Osier (2009) showed that the influence function of a quantile $Q(p)$ is

$$I(Q(p))_i = \frac{p - \mathbb{1}(y_i \leq Q(p))}{F'(Q(p))N} \, . \tag{9}$$

As explained by Van der Vaart (2000, p. 294), $I(Q(p))_i$ is a step function that takes two distinct values depending on the indicator function. Specifically, if y_i is below $Q(p)$, $I(Q(p))_i$ will be negative (indicating a decrease in the quantile), and if y_i is above $Q(p)$, $I(Q(p))_i$ will be positive (indicating an increase in the quantile), regardless of the distance between y_i and $Q(p)$. The magnitude of this influence is inversely related to $F'(Q(p))$, meaning that the quantile is less sensitive in regions where the density is higher.

The QRI influence function leverages the robustness properties of quantiles against outliers, since it depends on y_i only through $I(Q(p))_i$. $I(QRI)_i$ is obtained using the standard differential calculus rules (Osier 2009):

$$\begin{aligned} I(QRI)_i &= I\left(1 - \int_0^1 \frac{Q(p/2)}{Q(1-p/2)} dp\right)_i \\ &= -\int_0^1 \frac{\left(\dfrac{\frac{p}{2} - \mathbb{1}(y_i \leq Q(p/2))}{F'(Q(p/2))N}\right) Q(1-p/2) - \left(\dfrac{(1-\frac{p}{2}) - \mathbb{1}(y_i \leq Q(1-p/2))}{F'(Q(1-p/2))N}\right) Q(p/2)}{Q(1-p/2)^2} dp. \end{aligned} \tag{10}$$

The z_i resulting from (10) depends on the quantiles and the derivative of the cdf, which are unknown and must be estimated. By plugging their estimates into (10) we obtain

$$\widehat{z}_i = -\int_0^1 \frac{\left(\frac{\frac{p}{2} - \mathbb{1}(y_i \le \widehat{Q}(p/2))}{\widehat{F}'(\widehat{Q}(p/2))\widehat{N}}\right)\widehat{Q}(1-p/2) - \left(\frac{(1-\frac{p}{2}) - \mathbb{1}(y_i \le \widehat{Q}(1-p/2))}{\widehat{F}'(\widehat{Q}(1-p/2))\widehat{N}}\right)\widehat{Q}(p/2)}{\widehat{Q}(1-p/2)^2}\, dp\,. \tag{11}$$

The estimation of the income density function requires particular attention. It is the derivative of the distribution function, which in finite population theory is a step function, c.f. (2). Nevertheless, typically density estimation techniques are applied that smooth the empirical counterpart, for instance with the Gaussian kernel and a given bandwidth h,

$$\widehat{f}(y) = \frac{1}{\widehat{N}}\sum_{j\in s} w_j K\left(\frac{y-y_j}{h}\right) = \frac{1}{\widehat{N}}\frac{1}{h\sqrt{2\pi}}\sum_{j\in s} w_j \exp\ > \left\{ -\frac{(y-y_j)^2}{2h^2} > \right\}.$$

This estimator is used as suggested by Deville (1999) and Osier (2009). When data comes from a heavily skewed distribution, Verma and Betti (2011) suggests $h = 0.79 \cdot \mathrm{IQR} \cdot \widehat{N}^{-1/5}$, where IQR is the interquartile range. We employed this bandwidth suggestion.

In figure 2, the linearized variable z_i of the QRI has been estimated using the quantile estimator $\widehat{Q}_6(p)$, as an example, on 234 observations randomly drawn from a Log-Normal distribution with location and scale parameters 9.2 and 0.95. We see that the values of the influence function, $\widehat{z}_i$, stay in a small range close to 0. The influence function is a discontinuous function with a breaking point at the median (vertical blue line). Moreover, it is approximately constant for incomes above 50,000, as suggested by the boundedness of the influence function.

3. STATISTICAL BEHAVIOR OF THE QRI ESTIMATOR

This section focuses on exploring further characteristics of the QRI estimator, such as bias, error, distribution, sampling variance, and sensitivity to extreme

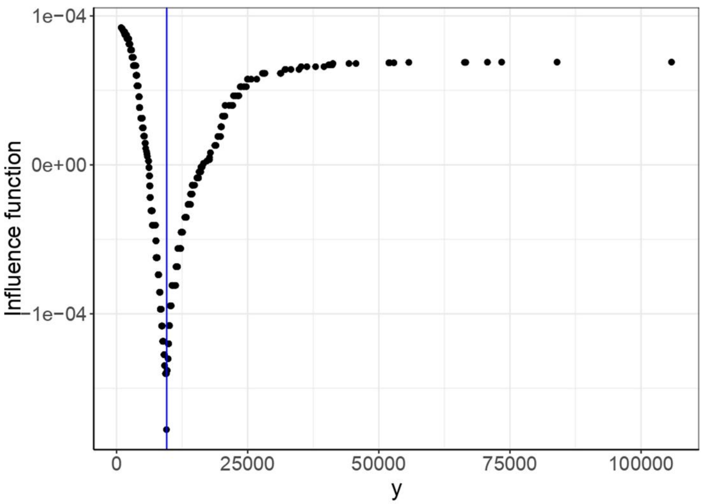

Figure 2. QRI influence function estimates against 234 observations drawn from a log-normal distribution with parameters 9.2 and 0.95 (blue line represents the median).

values. As Italian EU-SILC data are used in our simulations, we start with their brief introduction.

3.1 The Italian EU-SILC Data

The Italian EU-SILC survey employs a two-stage stratified sampling method. The Primary Sampling Units (PSU) are the municipalities, stratified according to geographical and demographic characteristics. The Secondary Sampling Units (SSUs) are the households. In this survey, municipalities distinguish between Self-Representatives (SRs) and Not-Self-Representatives (NSRs). The former are larger and considered as strata themselves. For NSRs, a two-stage stratified sampling design is performed: first, they are selected with inclusion probabilities proportional to the PSU dimension; afterward, a simple random sampling without replacement of SSUs is used across the selected PSUs. Further information about the survey can be found in the PRICSSA item checklist (Seidenberg et al. 2023). The main variable of interest is the equivalized disposable income, which has a heavily skewed distribution.

The data set from the IT-SILC 2017 survey is composed of 48,446 individuals and 22,226 households. These data are used to perform simulations based on synthetic and real data. As quality measures for our estimators, say $\widehat{\theta}$ of a population value θ, we use the relative and absolute relative biases defined as

$$\mathrm{RB}(\widehat{\theta}) = \frac{1}{R}\sum_{r=1}^{R}\left(\frac{\widehat{\theta}_r}{\theta} - 1\right), \quad \mathrm{ARB}(\widehat{\theta}) = \frac{1}{R}\sum_{r=1}^{R} > \left|\frac{\widehat{\theta}_r}{\theta} - 1 >\right|, \tag{12}$$

where R is the number of iterations in the simulation, and $\widehat{\theta}_r$ the r-th estimate, $r = 1, \ldots, R$. Another measure is the relative root mean squared error,

$$\mathrm{RRMSE}(\widehat{\theta}) = \sqrt{\frac{1}{R}\sum_{r=1}^{R}\left(\frac{\widehat{\theta}_r}{\theta} - 1\right)^2}.$$

3.2 Simulation on Synthetic Data

For generating synthetic data, we first fit to the IT-SILC data on disposable income some commonly used distributions, such as log-normal, Pareto, and the generalized beta of the second kind (GB2). GB2 provides the best fit, with one scale parameter ($b = 4.46$) and three shape parameters ($a = 3.71$, $p = 4.46$, and $q = 0.89$).[1] To simulate a complex sampling design, we first generated a finite population of $N = 100\,000$ observations with the GB2 distribution and then applied a setting similar to the one proposed by Alfons et al. (2013) for drawing samples from this finite population. An auxiliary variable $d = (d_1, \ldots, d_N)$ was created to assign probability weights for the sampling from our population. More specifically, for F_Q being the cdf of our GB2, and $g = 100$ equally spaced values between 1 and 10, we generated for each of the N observations y_i of our population the weight

$$d_i = \begin{cases} 1, & y_i > F_Q^{-1}\left(\frac{g-1}{g}\right) \\ 10 - \frac{9}{g-1}j, & F_Q^{-1}\left(\frac{j}{g}\right) < y_i \leqslant F_Q^{-1}\left(\frac{j+1}{g}\right) \quad \text{for any } 1 \leqslant j < g-1, \\ 10, & y_i \leqslant F_Q^{-1}\left(\frac{1}{g}\right). \end{cases} \tag{13}$$

1. Goodness-of-fit of the income data to some probabilistic distributions

Distribution	AIC	BIC
Log-Normal	1108515	1108530
Pareto	1114696	1118360
GB2	1096530	1096570

Then the Midzuno's method for unequal probability sampling was used. In the spirit of Alfons et al. (2013), this sampling scheme, which uses a proportional-to-size approach, assigns higher inclusion probabilities to lower income values.

The generated samples were then used to study the different QRI estimators obtained from the different quantile estimators of Section 2.3. Let $\widehat{QRI}_l$ be the QRI estimator using quantile estimators $l = 4, \ldots, 9, HD, Laek$. Each of them is calculated on $R = 1\,000$ samples drawn from the N units, having a sample size ranging from $n = 5$ to 200.

As said, the QRI will be estimated on a grid with $M = 100$ throughout, since larger M did not give different results; see table 2, which summarizes the ARB distribution for the QRI estimator (4) with $\widehat{Q}_6$ and $M = 50, 100, 200, 500$, computed on our Midzuno samples. This finding did not change for other quantile estimators.

Figure 3 reports the relative bias and the relative root mean squared error for the QRI estimators. The QRI estimated with the Harrell-Davis quantile estimator $\widehat{Q}_{HD}(p)$ in (6) seems to perform better than the others with RB values close to 0 for any $n > 50$. The $\widehat{QRI}_{Laek}(p)$ shows a competitive performance, though it is based on a discontinuous function. Somewhat surprisingly, when looking at our simulations on quantile estimators, QRI estimators that use $\widehat{Q}_6(p)$ or $\widehat{Q}_4(p)$ slightly overestimate the QRI for $n > 40$.

3.3 Simulation on Real Data

Again we draw $R = 1\,000$ samples, now directly from the IT-SILC income data, trying to approximate the original sampling design conditional on the available information. Here, the twenty Italian regions are used as strata and considered as target domains. PSUs are distinguished as SR or NSR. Samples are drawn under three scenarios, approximating 20 percent, 5 percent, and 3 percent of the target dataset size, with PSUs and households selected at rates designed to achieve these proportions. The regions' average sample sizes vary for each of these three sampling rates as follows: between 160 and 1,100, 37

Table 2. ARB Distribution of $\widehat{QRI}_6$ Evaluated on a Grid of Size $M = 50, 100, 200, 500$

Grid size	Min.	Q (0.25)	Median	Mean	Q (0.75)	Max
M = 50	3.795	4.387	5.212	6.483	7.121	24.208
M = 100	3.794	4.386	5.211	6.483	7.120	24.207
M = 200	3.794	4.386	5.211	6.483	7.120	24.207
M = 500	3.794	4.386	5.211	6.483	7.120	24.207

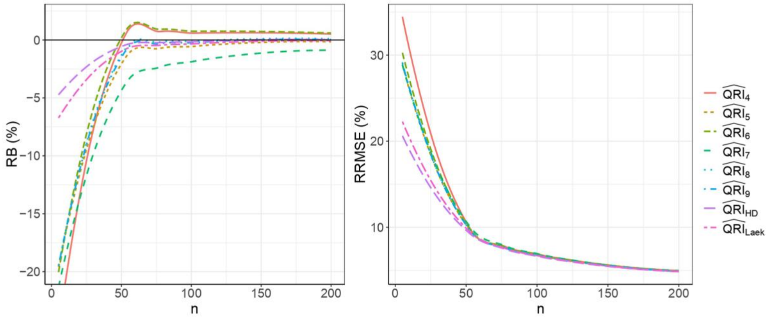

Figure 3. RB and RRMSE of our different QRI estimators.

Table 3. RB, ARB and RRMSE (in %) on Average of Estimates on 1,000 Simulated Samples from the IT-SILC Data with Different Sampling Rates

	20%			5%			3%		
	RB	ARB	RRMSE	RB	ARB	RRMSE	RB	ARB	RRMSE
$\widehat{QRI}_4$	−0.14	3.04	3.81	−1.53	7.76	9.74	−4.43	12.48	15.59
$\widehat{QRI}_5$	−0.21	3.05	3.82	−1.77	7.79	9.78	−4.80	12.50	15.63
$\widehat{QRI}_6$	−0.03	3.03	3.80	−1.02	7.70	9.65	−3.66	12.30	15.37
$\widehat{QRI}_7$	−0.38	3.07	3.84	−2.55	7.96	9.99	−6.11	12.89	16.10
$\widehat{QRI}_8$	−0.12	3.04	3.81	−1.50	7.75	9.73	−4.38	12.42	15.52
$\widehat{QRI}_9$	−0.16	3.04	3.81	−1.58	7.76	9.73	−4.50	12.44	15.52
$\widehat{QRI}_{HD}$	−0.18	3.04	3.80	−1.67	12.18	9.74	−4.66	12.58	15.67
$\widehat{QRI}_{Laek}$	−0.15	3.05	3.82	−4.50	7.78	9.76	−10.14	17.57	15.71

and 256, 21 and 134, with average numbers of observations being 485, 123 and 74, respectively. Based on these samples, we can study the quantile and QRI estimators for quite large, moderate, and small samples.

Table 3 reports the average RB, ARB and RRMSE in percentages over regions; results are given for each estimator and sampling rate. The RB values for $\widehat{QRI}_l$ are again close to 0 for samples of large to moderate size (20 percent sampling rate) and around -5% for small samples (3 percent sampling rate). Here, $\widehat{QRI}_6$ seems to perform best in terms of reliability, even for small samples. In any case, no matter which quantile estimator is used, the QRI estimators generally perform very well in the sense of having small biases and exhibiting moderate to small variances, at least for samples with $n > 30$. This might be related to the fact that the QRI does not depend on moments of distributions and considers the whole distribution via quantiles, which is particularly useful when relatively small or moderate sample sizes are used.

Our simulations on synthetic and real data sets demonstrate an excellent performance of our estimator. They also indicate the importance of a proper quantile estimator when estimating the QRI. Given the favorable results of the simulations on real data, we concentrate henceforth on the QRI estimator with the quantile estimator $\widehat{Q}_6(p)$, namely $\widehat{QRI}_6$ but skipping from now on the index $l = 6$. Note nevertheless, that further extensive simulations confirmed that the following findings also hold for QRI estimators based on other reasonable quantile estimators. Further numerical results are available on request.

3.4 The QRI Estimator Distribution

For subsequent inference, one needs the sampling variance and the distribution of the estimator. We therefore studied the $\widehat{QRI}$ distribution, using the estimates obtained from the above conducted simulations. Taking the results obtained on the real-data based simulations, we plot as an example the densities for the QRI estimates for Abruzzo, which is a medium-small domain among the twenty Italian regions. For the three considered sampling rates, we had on average 226, 72, and 37 observations for Abruzzo. The upper panel of figure 4 displays the densities obtained for the respective sampling rates; recall that the title indicates the average sample size, not a fixed one. The dashed red vertical lines coincide with the sampling mean and the blue lines with the population parameter. The dashed smooth green lines show the normal fits obtained from maximum likelihood estimates. The black lines are the kernel densities with Gaussian kernel and automatic bandwidth. The corresponding normal Q-Q plots in the lower panel of the same figure confirm that the observed quantiles hardly differ from the Normal ones. These results demonstrate that, with data collected through a complex sampling design, $\widehat{QRI}$ follows a normal law, even in small samples, with bounded support in $(0, 1)$.

3.5 The Sampling Variance of the QRI Estimator

As said, statistical inference on the QRI requires a feasible estimator for the standard error or the sampling variance. The QRI estimator is a highly non-linear complex function whose sampling variance can hardly be derived analytically. We therefore proposed and studied several methods for the estimation of the QRI sampling variance, but we limit our presentation to those that were most promising and easy to get. They can be grouped into the classes of linearization techniques and resampling methods, respectively. We start with the former one.

3.5.1 Linearization techniques.

As for the influence function, linearization techniques compute a linearized variable z_i associated with the statistic of interest for each unit i in the finite

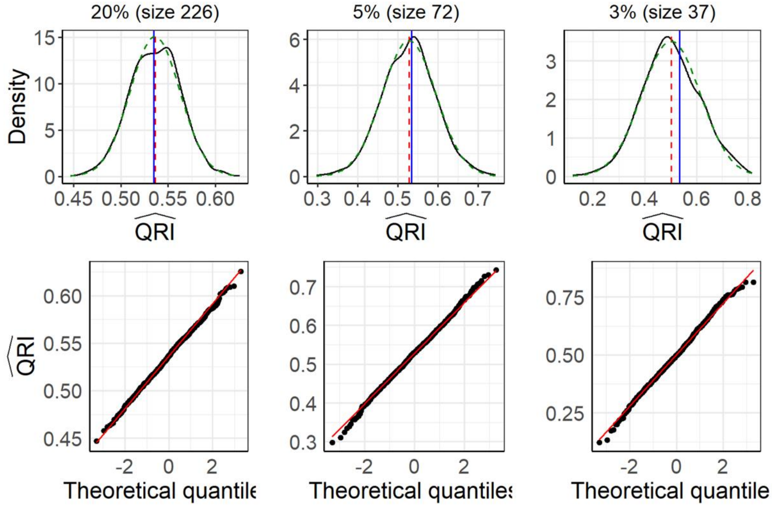

Figure 4. Empirical density (normal and kernel fit) and Q-Q Normal plot of the $\widehat{QRI}$ for Abruzzo (simulations based on IT-SILC data).

population U. They can be pretty helpful if the statistic of interest is rather complex, as is the case for the QRI. Specifically, the linearized variable can be used to approximate the sampling variance of the statistic. Unfortunately, the popular Taylor linearization method works only well for smooth functions of totals or means but cannot be applied to QRI, since it is a non-smooth statistic and not a regular function of estimated totals but only of quantiles. For those cases, Deville (1999) proposed a generalized linearization method based on the concept of influence functions.

The estimation of the variance of the total Horvitz–Thompson estimator $\widehat{Z} = \sum_{j\in s} w_j \widehat{z}_j$, where $\widehat{z}_j$ is computed through the influence function as in (11), can be used to approximate the $\widehat{QRI}$ sampling variance as

$$\widehat{\text{var}}(\widehat{QRI}) \approx \widehat{\text{var}}\Big(\sum_{j\in s} w_j \widehat{z}_j\Big). \tag{14}$$

As long as a proper expression for the variance estimator of the total estimator is known, (14) results in an analytical formula that can be applied to any complex sampling design. There exists an extensive literature on how to obtain estimators for (14); see Fuller (2011).

Unfortunately, for the QRI case, several problems arise. First, the linearized variable z_i has to be computed for any observation in the sample, which might be computationally intensive. Second, the QRI influence function is based on the integration of a very complex formula that requires the estimation of the quantiles and of the density function. For the latter, several methods exist, which may lead to slightly different results. To avoid this problem, Berger and Skinner (2003) proposed an influence function expression for the low-income proportion that does not include the density. Francisco and Fuller (1991) used Woodruff's confidence interval for quantiles to derive the variance estimator in stratified samples. It consists of inverting the confidence intervals of the empirical distribution, making the density estimation unnecessary. Yet, it turned out that the density function in the QRI influence function is not negligible. Furthermore, the boundedness of the influence function implies that the values of $\widehat{z}_j$ move in a small range; see figure 2, which results in an underestimation of the variance. This is also confirmed by the simulations reported in the next section. This was the reason why we also proposed and studied resampling methods.

3.5.2 Resampling methods.

Most of the popular resampling methods for estimating sampling variances in complex sampling are based on bootstrap; other methods are jackknife and balanced repeated replication. Bootstrap is very effective because it is applicable to both smooth and non-smooth statistics under general designs. The naive bootstrap method is not appropriate for finite populations because it does not account for the sampling design and assumes that the underlying population is infinite. This typically leads to a serious underestimation of the sampling variance. In the finite population literature, there exist many different algorithms that could be applied to complex sampling designs; Chauvet (2007), Wolter (2007), and Mashreghi et al. (2016) provide an exhaustive review. In this subsection, we focus on the literature on resampling algorithms that considers bootstrap methods, which were already applied to non-smooth functions. In the next paragraph we test these for the estimation of the sampling variance of $\widehat{QRI}$.

First consider the so-called superpopulation bootstrap method of Särndal et al. (2003). It consists of replicating each observation in the sample as many times as the corresponding sampling weight suggests, after calibration to some known totals. Calibration of the sampling weights can be performed with one of the methods reviewed by Tillé (2020); here we employ the raking method to calibrate the sampling weights to regional totals of age and gender. This way a (super)population $\mathcal{U}$ is generated. Then B bootstrap samples are drawn from it using a two-stage stratified sampling scheme. For each sample, the estimator of interest is computed the same way as for the original sample, and the empirical variance of these B estimates gives our variance estimator.

Another compelling method is the rescaling bootstrap, originally proposed by Rao and Wu (1988) for stratified sampling without replacement and functions of means. Consistency for non-smooth functions, such as our quantiles, was proven by Shao and Chen (1998); see also Kovar et al. (1988). Rao et al. (1992) extended the rescaling bootstrap method to complex sampling designs by proposing a rescaling of the bootstrap sample weights, particularly suited for multistage stratified sampling. In each bootstrap sample b, their procedure involves drawing with replacement m_h PSUs from the n_h ones contained in stratum h of the survey, such that $m_h \leq n_h$. Then, the original sampling weights w_{hlj}, referring to the individual j in the l-th PSU of stratum h, $l = 1, \ldots, m_h$, are adjusted using the following rescaling formula:

$$w_{hlj}^{(*b)} = > \left\{ 1 - m_h^{1/2}(n_h - 1)^{-1/2} + m_h^{1/2}(n_h - 1)^{-1/2} \frac{n_h}{m_h} m_{hl}^{(*b)} > \right\} w_{hlj}. \quad (15)$$

Here, $m_h = \frac{(n_h - 2)^2}{(n_h - 1)}$ as in Kolenikov (2010), and $m_{hl}^{(*b)}$ defines the number of times the l-th PSU is selected in bootstrap sample b, such that $m_h = \sum_l m_{hl}^{(*b)}$. The statistic of interest is estimated using these weights, and its variance is approximated by the bootstrap variance. This method is particularly useful when the original sampling weights were calibrated and adjusted to account for post-stratification and non-response, as is the case for our SILC data.

3.5.3 Comparing different sampling variance estimators.

To compare the estimation methods for the sampling variance of $\widehat{QRI}$ we took the $R = 1\,000$ samples drawn from the IT-SILC data with a sampling rate of 20 percent as described in Section 3.3. In addition to the above described bootstrap methods, we also studied the naive bootstrap, its calibrated extension (Templ and Alfons 2011), and the linearization method based on the influence function. The QRI variance was calculated by Monte Carlo methods for all 20 Italian regions, with summary statistics reported in table 4. Obviously, we are trying to estimate very small values, so somewhat large values for the relative errors could be expected.

Each bootstrap method used $B = 200$ bootstrap samples. Relative biases (RB) are reported in table 5. The linearization method based on the concept of influence functions does not work at all but always leads to a strong underestimation. The rescaling bootstrap of Rao et al. (1992) for multistage stratified sampling designs performs better than the other resampling methods considered. It never suffers from underestimation and approximates reasonably well the variance of $\widehat{QRI}$. Additionally, this method allows for the consideration and adjustment of the original sampling weights, which can be crucial when working with survey data. The weights typically carry important information, often being calibrated and adjusted for non-responses, as is the case with our

Table 4. Summary Statistics of the QRI Monte Carlo Variances of the 20 Italian Regions

Min.	Q (0.25)	Median	Mean	Q (0.75)	Max
0.00015	0.00025	0.00041	0.00044	0.00064	0.00090

Table 5. RB and Number of Negative RB Over 20 Regions for Different Bootstrap Methods

Bootstrap method	Min.	Q (0.25)	Median	Mean	Q (0.75)	Max	Nr.RB < 0
Linearization	−100.00	−100.00	−100.00	−100.00	−100.00	−100.00	20
Naive	−30.47	−25.95	−18.63	−17.37	−11.91	17.96	19
Calibrated	2.67	8.54	29.67	41.43	67.47	115.52	1
Superpopulation	88.65	125.28	144.91	149.41	165.29	238.66	0
Rescaled	19.13	31.44	41.37	42.05	47.40	78.74	0

IT-SILC data. Kolenikov (2010), Mashreghi et al. (2016) demonstrate that the rescaled bootstrap yields valid sampling variance estimates, without requiring replication of the original sampling design. This flexibility is very helpful in practice, as survey data often come from complex designs that are difficult to replicate exactly due to limited information and computational burden. Finally, our simulations indicate that the method works reasonably well even though the survey distinguishes between SR and NSR, a distinction that is ignored in the bootstrap procedure.

4. DISCUSSION

The measurement of inequality is a complex but by policymakers highly demanded task for National Statistical Offices. In addition, inequality indicators are frequently used tools in social sciences and economics. They serve decision makers to design or adjust their policies, projects, resource allocation, etc. Providing indicators that are reliable, even in samples of small and moderate size, is therefore of fundamental interest.

We consider the recently proposed quantile ratio index (QRI), introduce a modified version appropriate for finite populations, propose a nonparametric estimator for survey data, and study its performance under the use of different quantile estimators. We can show that our proposal(s) meet the request for reliability (small bias, small variance, and robustness) even for small samples and complex designs. As the QRI estimator needs to be based on an appropriate quantile estimator, we also study a series of natural candidates for them. It turned out that our QRI estimator provides an almost unbiased and robust statistical tool.

Finally, different estimators for the sampling variance or standard error of our QRI estimator are introduced, implemented, analyzed and compared. It turned out that the sampling variances can be estimated best by using a rescaling bootstrap method, which also accounts for the original survey sampling weights and thus ensures the preserving of the sampling scheme information. We also tried other methods like the generalized linearization methods based on the concept of influence functions. However, they seem not to work here. Nonetheless, the study of the influence function itself revealed some desired properties like boundedness, confirming the robustness properties and limited bias and standard error of the QRI estimate.

SUPPLEMENTARY MATERIALS

Supplementary materials are available online at academic.oup.com/jssam

DATA AVAILABILITY

The data underlying this article were provided by Eurostat under license/by permission. Data will be shared on request to the corresponding author with permission of Eurostat.

REFERENCES

Alfons, A., and Templ, M. (2012), "Estimation of Social Exclusion Indicators from Complex Surveys: The R Package Laeken," *Journal of Statistical Software*, 54, 1–25.

Alfons, A., Templ, M., and Filzmoser, P. (2013), "Robust Estimation of Economic Indicators from Survey Samples Based on Pareto Tail Modelling," *Journal of the Royal Statistical Society Series C*, 62, 271–286.

Berger, Y. G., and Skinner, C. J. (2003), "Variance Estimation for a Low Income Proportion," *Journal of the Royal Statistical Society Series C*, 52, 457–468.

Chauvet, G. (2007), *Méthodes de bootstrap en population finie* [Doctoral dissertation, EN-SAI].

Cowell, F. A., and Victoria-Feser, M. P. (1996), "Robustness Properties of Inequality Measures," *Econometrica*, 64, 77–101.

Davydov, Y., and Greselin, F. (2020), "Comparisons between Poorest and Richest to Measure Inequality," *Sociological Methods & Research*, 49, 526–561.

Deville, J.-C. (1999), "Variance Estimation for Complex Statistics and Estimators: Linearization and Residual Techniques," *Survey Methodology*, 25, 193–204.

Dielman, T., Lowry, C., and Pfaffenberger, R. (1994), "A Comparison of Quantile Estimators," *Communications in Statistics-Simulation and Computation*, 23, 355–371.

Farcomeni, A., and Geraci, M. (2024), "Quantile Ratio Regression," *Statistics and Computing*, 34, 94.

Francisco, C. A. (1987), *Estimation of quantiles and the interquartile range in complex surveys*, Doctoral dissertation, Iowa State University.

Francisco, C. A., and Fuller, W. A. (1991), "Quantile Estimation with a Complex Survey Design," *The Annals of Statistics*, 19, 454–469.

Fuller, W. A. (2011), *Sampling Statistics*, Hoboken, NJ: John Wiley & Sons.

Gastwirth, J. L. (2017), "Is the Gini Index of Inequality Overly Sensitive to Changes in the Middle of the Income Distribution?," *Statistics and Public Policy*, 4, 1–11.

Gini, C. (1914), "Sulla Misura Della Concentrazione e Della Variabilità Dei Caratteri", *Atti Del Reale Istituto Veneto di Scienze Lettere ed Arti*, 73, 1203–1248.

Giorgi, G. M., and Gigliarano, C. (2017), "The Gini Concentration Index: A Review of the Inference Literature," *Journal of Economic Surveys*, 31, 1130–1148.

Hampel, F. R. (1974), "The Influence Curve and Its Role in Robust Estimation," *Journal of the American Statistical Association*, 69, 383–393.

Harrell, F. E., and Davis, C. E. (1982), "A New Distribution-Free Quantile Estimator," *Biometrika*, 69, 635–640.

Hyndman, R. J., and Fan, Y. (1996), "Sample Quantiles in Statistical Packages," *The American Statistician*, 50, 361–365.
Kovar, J., Rao, J., and Wu, C. (1988), "Bootstrap and Other Methods to Measure Errors in Survey Estimates," *Canadian Journal of Statistics*, 16, 25–45.
Kolenikov, S. (2010), "Resampling Variance Estimation for Complex Survey Data," *The Stata Journal*, 10, 165–199.
Kreutzmann, A. K. (2018), "Estimation of Sample Quantiles: Challenges and Issues in the Context of Income and Wealth Distributions," *AStA Wirtschafts-Und Sozialstatistisches Archiv*, 12, 245–270.
Lumley, T. (2011), *Complex Surveys: A Guide to Analysis Using R*, Hoboken, NJ: John Wiley & Sons.
Mashreghi, Z., Haziza, D., and Léger, C. (2016), "A Survey of Bootstrap Methods in Finite Population Sampling," *Statistics Surveys*, 10, 1–52.
Osier, G. (2009), "Variance Estimation for Complex Indicators of Poverty and Inequality Using Linearization Techniques," *Survey Research Methods*, 3, 167–195.
Parrish, R. S. (1990), "Comparison of Quantile Estimators in Normal Sampling," *Biometrics*, 46, 247–257.
Prendergast, L. A., and Staudte, R. G. (2018), "A Simple and Effective Inequality Measure," *The American Statistician*, 72, 328–343.
———. (2019), "Decomposing the Quantile Ratio Index with Applications to Australian Income and Wealth Data," *European Journal of Pure and Applied Mathematics*, 12, 689–708.
Rao, J., and Wu, C. (1988), "Resampling Inference with Complex Survey Data," *Journal of the American Statistical Association*, 83, 231–241.
Rao, J., Wu, C., and Yue, K. (1992), "Some Recent Work on Resampling Methods for Complex Surveys," *Survey Methodology*, 18, 209–217.
Särndal, C.-E., Swensson, B., and Wretman, J. (2003), *Model Assisted Survey Sampling*, New York, NY: Springer Science & Business Media.
Seidenberg, A. B., Moser, R. P., and West, B. T. (2023), "Preferred Reporting Items for Com- Plex Sample Survey Analysis (Pricssa)," *Journal of Survey Statistics and Methodology*, 11, 743–757.
Shao, J., and Chen, Y. (1998), "Bootstrapping Sample Quantiles Based on Complex Survey Data under Hot Deck Imputation," *Statistica Sinica*, 8, 1071–1085.
Templ, M., and Alfons, A. (2011), *Variance estimation of social inclusion indicators using the R package laeken* (tech. rep.), Research Report CS-2011-3, Department of Statistics and Probability Theory.
Tillé, Y. (2020), *Sampling and Estimation from Finite Populations*, Hoboken, NJ: Wiley.
Van der Vaart, A. W. (2000), *Asymptotic Statistics*, (Vol. 3), Cambridge, United Kingdom: Cambridge University Press.
Verma, V., and Betti, G. (2011), "Taylor Linearization Sampling Errors and Design Effects for Poverty Measures and Other Complex Statistics," *Journal of Applied Statistics*, 38, 1549–1576.
Wolter, K. M. (2007), *Introduction to Variance Estimation*, (Vol. 53), New York, NY: Springer.
Zenga, M. (2007), "Inequality Curve and Inequality Index Based on the Ratios between Lower and Upper Arithmetic Means," *Statistica & Applicazioni*, 5, 3–27.

Journal of Survey Statistics and Methodology, 2025, 00, 1–21
https://doi.org/10.1093/jssam/smaf024
Article